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Epistemological and Didactic Obstacles: the influence of teachers' beliefs on the conceptual education of students

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Abstract
This investigation addresses the topic of teachers' beliefs about mathematics, mathematics education and their expectations of this discipline as regards their students' learning of two specific topics: fractions and angles. International research in this field has produced many results, the first being a good definition of the concept of beliefs, and has taken the first steps towards understanding the impact of these beliefs on didactic activity.
This research is based principally on the classical Theory of Obstacles. Within this framework, our aim is to establish whether or not, beyond objective epistemological obstacles, teachers' beliefs create some misconceptions among students. So didactical obstacles, arising specifically from some beliefs of the teachers, must be added to the epistemological obstacles. This is also clearly reflected in teachers' beliefs regarding the knowledge that their students must acquire.

1. Theoretical framework of the research

In view of the complexity of the research, there are two distinct specific theoretical reference outlines:
A) one relating to beliefs and to changes in beliefs, as well as to teachers' expectations regarding their students;
B) a general one relating to misconceptions.

A) Beliefs, changes in beliefs, and associated aspects
Changes in teachers' beliefs regarding a subject - mathematical, epistemological, or didactic - always represent an issue which is not easy to confront, because, in some cases, they come into conflict with sensitive personal and professional aspects. However, as the current research shows, they have considerable didactic significance. It is, in fact, universally acknowledged that beliefs are important constituents of the set of

that which is known, given that they establish and condition this set, as noted by Schoenfeld (1983) more than twenty years ago.
For an example of the studies conducted in this sector, and for the vast range of specific bibliography analysed therein, see D'Amore, Fandiño Pinilla (2004), a work dedicated to the changes in beliefs in mathematics, epistemology, and didactics on the part of teachers (secondary school) in training programmes.
Since we plan to discuss the theme of beliefs, we consider it to be of some interest to state explicitly that we will utilise the following definitions of the following terms (D'Amore, Fandiño Pinilla, 2004):
«*belief* (conviction): opinion, a set of judgments/expectations, that which one thinks about something;
someone's set of beliefs (A) about something (T) gives the *conception* (K) of A relative to T. If A belongs to a social group (S) and shares that set of beliefs relative to T with the other members of S, then K is the conception of S relative to T. Instead of 'the conception of A relative to T' we often speak of 'the image that A has of T'».
The shift from a personal belief to a shared conviction is extremely specific, given that beliefs are conditioned by complex interactions within social groups. In fact, since we are unable to separate the analysis of individual beliefs from the analysis of beliefs of the group one belongs to (Hoyles, 1992), we must also consider the micro-social aspect, which is very important in this research, as we will demonstrate.
That personal beliefs influence didactic and methodological choices has already been demonstrated in a work by the Research Nucleus of the University of Bologna, dedicated to the theme "area and perimeter" (D'Amore, Fandiño Pinilla, 2005). In this study, the personal beliefs of researchers, teachers, and pupils were placed in close relationship.
As highlighted by the research, it is true that beliefs can have deleterious effects on didactic action, but the opposite can also be true, as the results of our research show. We are also encouraged in this by the following statement "Beliefs can be an obstacle, but also a powerful force which allows the carrying out of changes in teaching." (Tirosh, Graeber, 2003). Moreover, in our work, great epistemological importance is placed on the socio-cultural perspective and on the interpretation of "practice" as constituents of the processes of conceptual construction, in a clear anti-Platonism, or even anti-realism view.
On the interpretation of the socio-cultural perspective, see the D'Amore, Radford, and Bagni interview-conversation (2006). On classroom, mathematical activity, see D'Amore, Godino (2006). On the realism – pragmatism dichotomy, see D'Amore, Fandiño Pinilla (2001), D'Amore (2003), D'Amore, Godino (2006). This last work puts forward a historical-critical analysis of the derived perspectives; both anthropological and ontosemiotic.

B) Misconceptions
One of the first documented appearances of the term "misconception" in mathematics occurred in the USA in 1981, in a text by Wagner (1981) which treated the learning of equations and functions. Again in 1981, a celebrated text by Kieran (1981) discussed equation solution activity. 1982 saw the publication of an article belonging to the algebra learning domain: Clement (1982). In 1983 we have works by Wagner (1983) and Kieran (1983), again on algebra. Then numerous works published in 1985 specify the term "misconception": Schoenfeld (1985), Shaughnessy (1985) and Silver (1985), who use it mainly with regard to *problem solving*, together with beliefs or to explain their interactions.

In Silver (1985, pp. 255-256) it is stated explicitly that there is a strong tie between misconceptions and mistaken beliefs.
Schoenfeld (1985, p. 368) highlights how students can correctly develop some incorrect conceptions, particularly regarding procedures.
As we can well see, in the first half of the 80s intense work on this theme was conducted by Mathematics Education scholars. Fischbein himself, in the 80s and 90s worked in this field, only sometimes, however, using the term explicitly, particularly in relation to the learning of probability (Fischbein et al., 1991; Lecoutre, Fischbein, 1998).
Interesting citations of the term appear in Furinghetti, Paola (1991) and in Bonotto (1992), where it is given as a synonym for “incorrect rule” (p. 420). The term is used several times in Arzarello, Bazzini, Chiappini (1994) with regard to the learning of algebra. In all of these works, the term is interpreted in the negative sense acquired from the literature.
The term “misconception” is also used considerably in Gagatsis (2003), with the same meaning.
Bazzini (1995) maintains: «In the realm of the most recent studies, a fascinating area of investigation is the one relative to the function of analogical reasoning in the process of the restructuring of individual knowledge and of the overcoming of misconceptions (Brown, Clement, 1989)».
Further on, citing Fischbein, she reports: «We should not, however, forget that if different kinds of analogical reasoning on the one hand can encourage the construction of knowledge, on the other they can induce erroneous conclusions at the moment in which particular aspects are emphasized or distorted to the disadvantage of others. If the analogy is a potential generator of hypotheses, it can also cause misconceptions or misunderstandings (Fischbein, 1987, 1989). It often happens that when the subject finds himself in considerable uncertainty, when facing a problem to be solved, he is brought to transform a certain nucleus of information from a well known domain to another less known one by means of a transfer by analogy. It can happen, then, that one can assume analogical correspondences to be valid which, instead, are not plausible for those particular systems. We are speaking of tacit analogies which can insert themselves into the cognitive process and disturb it».
The concept of misconception was not precisely defined at the moment of its entrance into the world of Mathematics Education research, but, as we have seen, it was, and still is, used with its intuitive meaning.
Having taken into account the positions of the different authors and the sometimes rather diverse occurrences of this term, we maintain that the attention given to misconceptions, since their appearance in the world of the sciences (not mathematical), has been very productive because it has forced scholars to no longer identify errors as something absolutely negative, to be avoided at all costs, but as human products caused by situations on the path of evolution. Increasingly often, over the years, a shared meaning of “misconceptions” has been delineated as causes of errors or, more precisely *perceived* causes of errors; causes which are often very justifiable and sometimes even convincing.

It is therefore undeniable that this kind of study has forced us to examine the interpretation of the reality of the subject; an interpretation created on the basis of convictions which have matured partly as a result of learning; therefore, to view misconceptions as the fruit of something known, not as an absolute lack of knowledge. (D'Amore, Sbaragli, 2005).

In recent years, we have also been starting up a first classification of misconceptions, observing their specific features. A first distinction regards those which we have called avoidable and unavoidable misconceptions (Sbaragli, 2005).

Avoidable misconceptions derive *directly from the didactic transposition of knowledge and from didactic engineering,* in that they are, precisely, a direct consequence of the choices of teachers. These misconceptions arise from scholastic practices "undermined" by improper habits proposed by teachers to their pupils.

In fact, it often happens that decisions taken by teachers complicate the learning of mathematical concepts: decisions, sometimes deriving from the proposals of the noosphere (textbooks, programmes, magazines,…), to supply to the pupil, day after day, always and only unambiguous conventional representations which are in this way blindly accepted by the pupil because of the *didactic contract* established in class and the phenomenon of *scholarization* (D'Amore, 1999a).

The continuous and unambiguous demands by the teacher ensure that the student, and even sometimes the teacher himself, confuses the proposed representation with the mathematical concept that he wants to teach: «The student doesn't know that he is learning signs that stand for concepts and that he should instead be learning concepts. If the teacher has never reflected on this point, he will think that the student is learning concepts, while, in reality, they are "learning" only to use signs» (D'Amore, 2003).

The repetitiveness of the representations supplied is not the only cause of avoidable misconceptions, which also often depend on representations that have been badly chosen by the teacher himself (Martini, Sbaragli, 2005; D'Amore, Fandiño Pinilla, Marazzani, Sbaragli, 2008).

The unavoidable misconceptions are those which derive only *indirectly from the choices* made by the teacher, in that they are a consequence of the need of having to say and prove something of a non-definitive nature in order to explain a concept.

Such misconceptions can therefore be attributed to the necessity of having to start from a certain knowledge in order to be able to communicate; an initial knowledge that, in general, will not completely encompass the entire mathematical concept that one wants to propose.

At this point, we can find a connection between the two analysis elements of the knowledge construction process. The ontogenetic and epistemological obstacles seem to be tied to the idea of unavoidable misconceptions, given that they depend both on the maturity of the pupil in his ability to conceive a specific piece of mathematical knowledge (ontogenetic obstacle) and on the concept itself, often epistemologically complex (epistemological obstacle) which is proposed. At the same time, the avoidable misconceptions are tied to the idea of didactic obstacles depending on didactic transposition and on the didactic engineering choices made by the teacher.

It is, therefore, the task of the teacher to pay close attention to misconceptions and to the obstacles which can arise during the teaching-learning process; to be aware that what the student thinks are correct conceptions can in reality be misconceptions and that the cause of these misconceptions can depend on different types of obstacles. So our research focuses strongly on the didactic aspects. Obstacles and misconceptions are therefore strongly connected, particularly if these issues are viewed from the aspect of the problems encountered by the pupil in terms of mathematical conceptualisation.

2. Research questions and preliminary hypotheses

The research questions are listed here below.
Q1. What beliefs do elementary and secondary school teachers hold about some aspects concerning the subjects of fractions and angles, from both the cognitive and the didactic aspects?
Q2. What are the expectations of teachers, at this scholastic level, regarding the performance of their pupils relative to the two mathematical topics in question?
Q3. What are the beliefs of pupils, aged 9 to 14, about these aspects concerning the topics of fractions and angles? How do these beliefs evolve over the years?
Q4. Is there coherence between the expectations of teachers regarding the performance of their pupils and the results effectively achieved during tests?
Q5. Do the beliefs of the teachers referred to in Q1 affect those of the pupils referred to in Q3? In what way? Is there relevance amongst the results obtained between the two samples in question?
Q6. On the basis of the answers obtained in Q1-Q5, can we hypothesise that the epistemological obstacles related to the subjects in question (angle and fraction) are not solely responsible for the potential cognitive failure of students, which can also be attributed to didactic obstacles?

Listed here below are the respective preliminary hypotheses formulated at the beginning of the research and thereafter confirmed or proven wrong by the results of this same research.

H1. Although all teachers will have completed a teacher training programme, we hypothesised that some teachers (often, but not only, primary school teachers), would have demonstrated some uncertainty in the implicit mathematical formulation in our proposals, particularly because of the transition, testified by international research, linking academic Knowledge, acquired in the years of study, to the school practice of didactic transposition, which, in the end, also conditions the knowledge of the teacher. In our opinion, this also implies that the didactic convictions of teachers are "modelled" on the their cognitive expectations regarding their students.
H2. The considerations of H1 condition the expectations of teachers, creating the illusion that their cognitive proposals, if appropriate from an adult perspective, are easily acquired and constructed by students. We therefore assumed that teachers would give excessively positive preliminary opinions on the actual results of their students.
H3. We decided that the intentionally generic question would only be responded to after the tests. However, we hypothesised that the increasing age and the consequent increase in level of skill would not have excessively modified the knowledge acquired in elementary school.
H4. As already said in H2, we hypothesised an excess of positive pre-evaluation by the

teachers.
H5. We hypothesised that there would be close coincidence, easily identifiable by comparison, between on the one hand the answers to the questionnaires and the interviews done with pupils, and on one other hand, the interviews conducted at the same time with the relevant teachers, on the other.
H6. We hypothesised that, as noted by the international research in other sectors of mathematics, also in this case the undoubted existence of epistemological obstacles would have been supported by the creation of didactic obstacles, which would be easily noted on the basis of the answers given to the preceding research questions. In short, the distinction between the two categories of obstacles is foundationally correct, but perhaps didactically nonexistent.

3. Research methodology

The test was designed according to the following methodology.
Researcher 'A' gives a class a questionnaire on fractions and angles (see the attached document), with questions that are designed to highlight potential misconceptions.
At the same time, researcher 'B' interviews the teacher of the class, outside the classroom, on the same topics as those listed in the questionnaire given to the pupils.
When necessary, follow-up interviews are conducted with pupils to clear up possible ambiguities.
There is therefore no interference between the class teacher and the work of the pupils.
In academic year 2007/2008, pre-tests were conducted in 4 classes; 2 in the ES (elementary school) and 2 in the MS (middle school), on a total of about eighty pupils, to start calibrating the research topics and the methodological approach to be followed. The results of these initial investigations allowed us to clarify the feasibility of this research and to prepare the final versions of the tests.
According to the research plan, therefore in academic year 2008/09, 12 classes in Ticino Canton (Switzerland) were evaluated, 6 ES and 6 MS, giving a total of 105 ES pupils and 115 MS pupils.
The same tools were used for all tests:
two researchers A and B for each visit, with the same functions, but now equipped with an improved grid both for recording the interviews and for their subsequent analysis;
tape recorders were used during almost all the interviews;
students were given written tests consisting of both multiple choice questions and questions with semi-open answers; in part, on subjects already known and in part, on new subjects;
the interviews were carried out with the teachers at the same time as the questionnaires were given to the students. The technique was that of a clinical interview;
immediately after each class visit, the group met up to analyse the notes taken during the interview itself and to listen to the audio recordings.

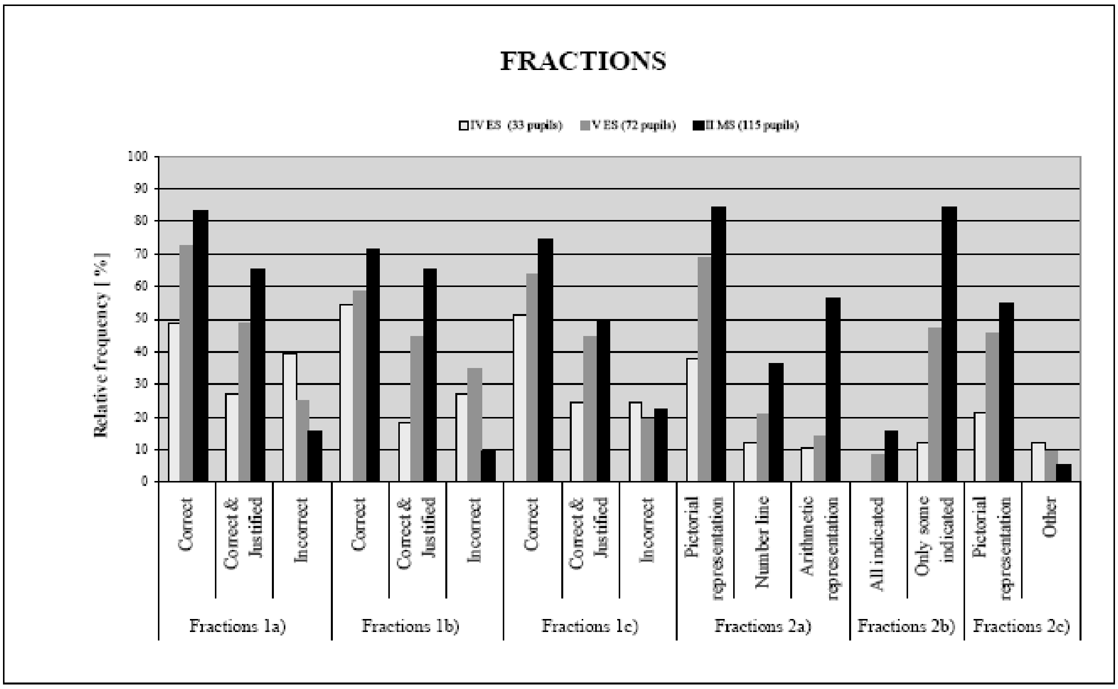
4. Initial results

The section below reports the results of the research, divided between the beliefs of the pupils and those of the teachers for both of the mathematical contents subjected to this analysis: fractions and angles. We have chosen to highlight more the beliefs of the teachers, which allows us to give more exhaustive answers to our research questions.

4.1 General results of the pupils

The section below reports the results of the pupils obtained from the questionnaire and from the subsequent interviews.

4.1.1 Fractions



We will begin with the most predictable results. Globally, more pupils answer the questions correctly than make mistakes. If, however, as also transpires from the expectations of the teachers, we consider only the correct and justified answers, the global result is much less satisfactory. Comparing these results in the various classes (4^{th} and 5^{th} ES, 2^{nd} MS) we note that increased scholastic level is associated with constant progress in all the items.

Amongst those who did not respond correctly, the following error typologies, encountered at both scholastic levels, were highlighted; students:

went astray on the non-congruent parts: “No, because they are not divided into equal parts” (5^{th} ES);

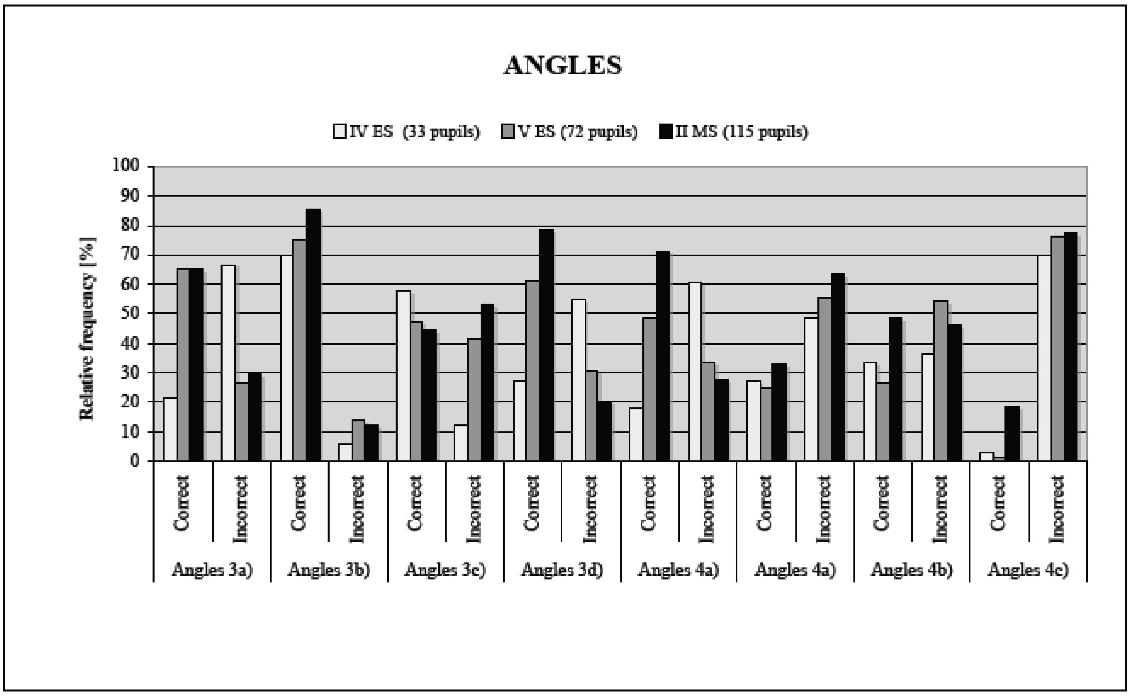
considered only half of the figure: “No, because they are divided in two” (5^{th} ES);

couldn’t conceive that two distinct parts could constitute one single part: “No, because neither of the parts is near or attached” (2^{nd} MS);

didn’t know how to arithmetically manage the fractions: “No, because it doesn’t represent the ½ part, but 2/4” (5^{th} ES); “Because part A is ¼ and part C is also ¼ and in

all it is 1/8" (2nd MS).
It should be noted that there was a significant difference between the answers to questions 1a and 1b, which makes us hypothesise that the form of the figure affects the recognition of the corresponding fraction.
The pictorial register is considered (even by the MS pupils) the most appropriate of the semiotic registers for representing the fraction ¾. This could be ingenuously surprising considering that, in Ticino, it is in this scholastic period that the theme of fractions is addressed again, placing emphasis on the arithmetic register. We also noted some difficulty in recognizing the same meaning for different representations. Moreover, it should be noted that on several occasions the answers turned out to be contradictory. Some pupils may have read question 1d too quickly, and may have failed to consider the *not*. For example, one pupil (and not the only one) who overall responded well to all the fraction questions, wrote the following answers for question 1b), 1c), and 1d):
1b) 1 and 2 because they both make ¾. 3 and 6 because they both are 0.75.
1c) NO.
1d) 1 and 2.
The concept of fraction as a rational number is not greatly focused on in the ES in Ticino; here also, one can note an excessive fear on the part of the teachers.

4.1.2 Angles



The fact that the concept of angle as part of a plane is not generally tackled in the ES, leads us to believe that several unjustified answers, particularly of the 4th ES pupils, are casual. For example, for item 3c the pupils of the 4th ES were even better than both those of the 5th ES and those of the 2nd MS.
To answer questions 3a, 3d and 3e, a substantial number of the ES pupils used a goniometer (protractor), a strategy used in the classroom by many teachers, and this has made it even more complicated to interpret the results because small inaccuracies or errors in measurements have significantly influenced the percentage of success. This is not true for the pupils of the MS, who knew how to apply their knowledge of vertically opposite angles.

The answers to questions 3b and 3c have highlighted a very common kind of error, which consists of confusing the sign used to indicate the angle (the symbol) with the angle itself (the object): “It isn’t ‘inside’ the angle.” (2nd MS).
The lack of conceptual knowledge of the angle profoundly influenced their success in item 4, in which poor quality answers were encountered. At least three different interpretations of the idea of angle are seen in these incorrect answers:
point coinciding with the origin; “Because the angle has only one point, which is B” (2nd MS) or with the vertex of a polygon; in 4a) one pupil answered: “NO: because D is from another part” and justified points C and P in the same way;
union of segments: [D does not belong] “Because the two lines of B do not connect with the […] point” (2nd MS);
limited surface: “Because it is completely outside, even from the diagram” (2nd MS); the expression “inside angle” probably contributes to reinforcing this last view of angle coinciding with the polygon;
“arc”; confirming that for some pupils the angle is identified with the “arc” which is usually used to represent it, one pupil answered question 4a): “NO: because it has nothing to do with it and it’s too far away”; the same thing happened with question 4b).

4.2 Teachers’ results

4.2.1 Elementary school teachers

We report some considerations deriving from the interviews conducted with 6 elementary school teachers; this report is also divided between the subjects “fraction” and “angle”, accompanied by conversation extracts reinforcing the considerations.

4.2.1.1 Fractions

First situation - Conceptual aspect

In terms of the conceptual knowledge of the elementary school teachers on this piece of knowledge, the 6 subjects questioned they were not perceived to experience any particular difficulties in answering correctly the questions submitted to the pupils. Only one teacher demonstrated significant indecision in answering question 1c) and this indecision was due to the requirement to obtain only congruent parts amongst themselves when talking about fractions.
We report an excerpt of the interview from which this indecision emerged:[1]
1: You’ve put me in a difficult situation, for that initial one, [of the rectangle] I wouldn’t know, for that one under it, yes, it is half, but of that …
I.: Does all of this coloured part A + C together represent ½ of the total rectangle?
1: Yes, however there is still another half and another half of C which are white.
I.: But what are you looking at in this figure?

[1] We decided to number the teachers and to indicate the interviewer with I.

1: I'm trying to report the figure here.
I.: But what about these figures: look at the shape, the perimeter, …
1: The problem is that if there are *two different figures*… No, in the end nothing changes if this little rectangle has the same dimensions as this one. But, I don't know.
I: You always refer to the first rectangle, but is it different if you refer to this rectangle? Does anything change?
1: No, because it's equal.
I.: Ok, then A and C together?
1: They make half.
I.: So, the difficulty depends on the different shapes?
1: Yes, for me yes, if C were also like this (the same shape as A), then it would jump out at me that there are 4 equal parts.

The questions asked were answered well by one teacher, but she emphasised that she considered the concept of fractions differently if talking about arithmetic or about geometry, not realising that the fractions proposed are associated with plane figures in a continuum and therefore refer to areas. So, in the situations proposed to the teachers and to the pupils, the arithmetic aspect turned out to be profoundly connected to the geometric aspect.
2: This is a conversation [questions about the rectangle] which, for example, amongst other things, "If you speak about areas, it definitely works, even if I then say that it has nothing to do with fractions. Let's say that the fact of transporting a part of the figure in order to understand, for example, I don't know, the area of the parallelogram, I transform it into a rectangle, moving a triangle, I don't know if I'm explaining myself well, eh …. Surely, doing activities more… they will certainly get more used to seeing, even if I repeat that this is not important with fractions, I don't want to create confusion with fractions at this point. But, when you do activities with young people, such as I don't know, geometry, I think they use those activities in other environments, it isn't that they don't use it, but without confusing the subjects. I don't want you to think that I'm confusing the areas with the fractions".

First situation - Didactic aspect

As concerns the didactic aspect with reference to the subject of fractions, 5 out of 6 teachers interviewed maintained that it is indispensible to show congruent parts to their pupils when speaking about fractions and they were not able to hypothesise how it would be possible to do otherwise in class, also because they are concerned about giving pupils problems that are too difficult for them; these hypotheses have not been proven in practice, however. These teachers turn out to be closely tied to the traditional practices in use in the Ticino Canton. Although recognising that, in the geometric register, the unit can be partitioned into non-congruent equi-extended parts, in 5 cases out of 6 examples of this kind had never been proposed in class: the teachers stated that they had never considered the problem.

I.: Do you ever give fractions of this kind where the figures are not congruent as in this case?
3: No, absolutely not, never.
I.: When do you tackle examples like these, where the parts are not equal, but represent the same fractions?

3: I don’t know when I will do it. I do fractions with DiMat.[2]
I.: Does DiMat cover cases of this kind?
3: No, they are not covered in DiMat, not even in the fifth year. When we present fractions, it is explained that they must be divided into equal parts, because, otherwise, they would never spontaneously divide them up in that way [meaning: in non-congruent parts].
I.: Does DiMat specify that making fractions means dividing into equal parts?
3: Yes.
I.: And, in your opinion is it right to say that making fractions means dividing into equal parts?
3: As an introduction, of course. Otherwise, they wouldn’t understand. They already find it difficult to understand fractions in this way. Just imagine, with this more difficult example, they wouldn’t understand. This could be a later example, but definitely not the initial one.

I.: Do you show examples where it is not necessary for the parts to be equal?
4: No, in my opinion they must divide into equal parts.

I.: Do you ever show examples like this?
1: Like this one, no. I’ve never shown it.
I.: Why not?
1: Well, I’ve never shown an inside rectangle divided in this way… that is, I have already shown, for example, a rectangle subdivided like this [all equal parts] divided in half, in tenths, but the same figure divided in two, no, I’ve never done it.
I.: You’ve never shown examples with differently-shaped parts?
1: Yes, that’s right.
I.: Do you use DiMat?
1: Yes.
I.: In DiMat, are there no situations like this?
1: There are a number of situations where they have to… where they have, for example, some squares where they have to colour 1/8, 1/10, or they have to subdivide some chocolate bars and then I have them construct a sheet of paper with a grid, then, for example, there are 4 bars and there are 6 children who have to divide them up. Last year I had a wonderful experience with fifth year pupils, which was given by the ‘school board’. So, I’ve never done it, but I am not so tied to the DiMat. I sometimes prefer doing lessons that are face-to-face, because it isn’t an easy subject. I often show them cakes where I divide them into 4 parts and I show them ¼, it is a bit real, like the slices. It’s difficult, also in DiMat. At F level they colour in, they understand it well, but already at M it is already difficult, even for the good students,[3] they don’t finish it,

[2] DiMat (Differenziazione dell’insegnamento della Matematica – Differentiation in the teaching of Mathematics) is a commonly used approach to the teaching of mathematics in Ticino in the 3rd – 4th – 5th elementary school classes. DiMat specifies the use of worksheets, which allow differentiated student learning paths.
[3] F and M are two levels of the DiMat.

precisely because it is complex. They know that 1/5 of 25 means divided by 5, however … it isn't obvious.

Only one teacher stated that he had recently worked (confirmation proven by the results of the class) on situations (graphic register) representing equi-extended, but not isometric, fractions. When dealing with these situations, the pupils of this class based their work on counting the squares and not on the relationship between the parts and the whole.

Comparison of expectations and class results

Three teachers stated that some children could be mislead by the fact that the shapes of the parts are not congruent and, as we have previously seen, this comes from their didactic practice which does not cover these cases. At different levels, all three of them overestimate the results of their pupils.

I.: In your opinion, will your pupils have problems?
1: Perhaps some, yes, they are not all… I think that only some will have problems.
In particular, teacher C overestimated the results for question 1b) where the correct answers were only 40% versus his forecast of "a few" who would make mistakes.

The second teacher thought that his pupils would answer well in part, but would not know how to explain the correct answer, the subject of fractions is generally rather complex.
5: They will say "yes", but they won't know why. They are generally quite unsure about fractions and they will then give random answers. But … I don't know how my pupils will answer. It is really difficult for them. Some could answer well, but I don't know if due to real knowledge, the others will answer at random without saying why. Not even the good ones will say why.
Although predicting some difficulties, this teacher:
I.: Where are the greatest difficulties with these questions?
5: As we were saying before, precisely because the parts are not equal, if it had been divided into 4 equal parts, it would have been easier for everyone to understand, but as it is …
5 completely overestimates the results. The results are, in fact, 17.6% for the first question and 5.9% for the following two. It should be remembered, however, that we're dealing with year four pupils.

The third teacher also overestimated the results, maintaining that most should answer well, while, in fact, only 35% answer 1a) correctly and 45% answer 1c) correctly.
2: I think that they should know it, but I think the fact that there is a triangle here and a rectangle there, could, instead, create a bit of confusion, because it is not always easy when there is a small complication within the situation.
…
2: I don't know if all, I don't know, some have difficulties, a substantial little group, 4-5 pupils who work at their level of calculation and automatism, but of reasoning no, but I'm a bit of a pessimist… the previous teacher, however, also said so, that it is not an easy class.

Instead, one teacher evaluated the abilities of his pupils well. In particular, the fact that revision and further study had recently been done on fractions allowed her to anticipate

that most of her pupils would answer the first situation questions satisfactorily. There is, however, an exception to question 1c). Only 55% (versus the expected 70-80%) stated that A and C together represent ½ of the whole figure, with pupils often giving explanations contradictory to those of the first two answers:
1a) Yes, because the two equal parts have been divided up;
1b) Yes, because one half has been divided;
1c) No, because, if not, it would be completely coloured.

One teacher was not off balance in his predictions, because most of the questions concerned subjects barely or not yet tackled.
Another teacher, for questions 1a), 1b) and 1c) predicted a success rate of 60% and broadly speaking reached this level.

Second situation – Conceptual aspect

As regards the second situation related to the subject of fractions, no particular conceptual difficulties were noted regarding recognition of the different representations as equi-meaningful to ¾, except for most of the cases which associate the representation 4 to 3/7, as was predictable and also correct. However, it should be noted that the context certainly made them pay more attention when giving the answers.

3: Yes, they are all definitely ¾.
I.: For you, are they equi-meaningful or not?
3: Yes.

2: 0.75 is 0.75; I would say 3, 5, and 6 say the same thing which is 0.75. And also the first and the second and this one here, left hanging [number 4] is left a bit on its own, although if it could be interesting. But, I see it as 3 over 7 not 3 over 4.
I.: Does each of these representations say the same thing or something different?
2: Perhaps yes … But, but in the end they say the same thing, but actually I don't like the mathematical expression very much, let's say it expresses the same quantity, that's a way of saying it.

I.: In your opinion, do all of these mean the same thing?
C.: This one no, because it is 3/7, the others are all ¾.

One teacher considers 0.75 as ¾ of 1 and not as an equi-meaningful representation of ¾ in itself.
2: 0.75 represents ¾ of 1.
I.: ¾ of 1?
2: Of 1, because 1:4 makes 0.25 times 3 makes 0.75.

The preference for the representation which better expresses the concept of ¾ is concentrated, more than others, on the first, in one case on the fifth and in one on the

third:
I.: Is there one of them which gives the idea of ¾ more than the others?
3: I prefer the first, because I now reason like my pupils.
I.: Why?
3: It's the one that, at first glance, gives more the image of ¾.

2: The first is the one I'm more used to for ¾, also the children.

I.: Is any one of these representations that is better than the others or not?
2: I like number 5, or the first one, even number 6 isn't bad, I like decimals.

1: I don't know, they are so different, perhaps this one [number 3] because it's more original. But, then, I see either 3 or 5 in fraction, otherwise I don't see the fraction.

Second situation – Didactic aspect

From a didactic point of view, the teachers state that they don't explicitly work on all these semiotic representations of the same concept, at most only on some. The impression is that, despite being quite inclined to admit the equi-meaningfulness of the representations (except in the ambiguous case of number 4), in class the teachers feel hampered by their concern about showing excessively complex examples. So, the use of various semiotic registers is willingly deliberately reduced and/or postponed for as long as possible.

1: I don't show this kind of activity in class.
2: I don't explicitly say that ¾ and 0.75 mean the same thing, at the most ¾ and the first drawing.

One teacher stated that he used only two registers: the graphic one (equi-extended areas) and the fractional one. Other registers were not presented to the pupils.
One teacher stated that he had only mentioned "without insisting" the fraction-decimal number passage and that he had not ever used the representation of rational numbers on the numeric straight line. Moreover, he maintained that the representation of the single interval [0.1] constitutes a substantial epistemological obstacle.
6: "The pupils are used to handling the numeric straight line with positive whole numbers. A few simple decimals have already been inserted, but it's quite another thing to deal with the rational interval only [0.1]".

Comparison of expectations and results of the class

In four cases, the forecasts, although negative, were better than the results. For example, for the first representation, regarding which a teacher stated he was certain the pupils would recognise the concept of ¾, 50% recognised it. A similar result was recorded for the fifth representation, which was predicted to be recognized by 50% of the pupils but which was actually recognised by only 20%. Another teacher also predicted a good recognition success rate for the first three representations, but this was not the case, and percentages of only 2.4%, 23.5%, and 17.6% were recorded. The third teacher predicted good recognition success for the first, the second, and the fifth representations, while the results were 20% for the first two and 0% for the fifth. One teacher predicted a success rate of at least 50% for the second and third representations, but the results were much lower.

Different semiotic registers are used with extreme prudence and particularly in the fifth year, so one teacher correctly predicted that the 4th year pupils (this was a multi-level class, with 10 year 4 pupils and 8 year 5 pupils) would limit themselves to discrete object subdivision and to geometric subdivision.
For the fifth year pupils, she evaluated her pupils' abilities well. She predicted that there would be satisfactory answers for representations 1 and 2, but considerable difficulty in recognising the fraction ¾ in the others. As, in fact, happened, she anticipated that many would see the fraction 3/7 and not ¾ in situation 4.

One teacher was not off balance in his predictions, because most of the questions concerned subjects barely or not yet tackled. When he abandoned caution, he made optimistic forecasts ["Everyone should answer this question, except for the usual ones…" questions 1a), 1b), and 1c)]. In fact, the class successfully answered around 50%.

According to the teachers interviewed, the pupils don't realise that the representations are equi-meaningful, since this item of knowledge had not been introduced in class.
I.: In your opinion, will your pupils realise that we are saying the same thing?
2: This is difficult, I don't think so, I'm a bit pessimistic, and I admit it.
I.: Do you think the children will match them as you did?
2: Not all of them, I don't think that all the children, perhaps 0.75 and 75/100 that yes, but match them, no, if only! It is difficult because, in certain things, it isn't so easy to know what they'll answer and what they'll not answer. In this class, you often have to revise for a small group.

Regarding the prediction of the choice that the children will make for the representation which best expresses ¾, everyone agrees that it will be the first:
3: The first in my opinion. The ones that will sidetrack them most will be numbers 4) and 6). Matching ¾ to 0.75 is not a logical combination.

1: Number 1, because it goes over material that they have seen and then there is this one figure and that's all. There is no doubt. Instead, here, you at least wonder, but what is the sack [he's speaking about number 2]? Thinking about DiMat, if each sack is a tenth and the unit is sometimes not very clear inside the sack.
I.: If you had to represent the concept of ¾, which representation would you suggest?
1: I would take a cake and divide it into 4 and I would say that we can eat 3 slices.
I.: So, like number 1?
1: Yes, with them, yes.

4.2.1.2 Angles

First situation – Conceptual aspect

Of the 6 teachers interviewed:
2 answered all the questions correctly;
1 stated a situation of uncertainty in question 3b), about whether point C belonged to the highlighted angle (a question related to whether the side is part of the angle or not);
1 demonstrated not having ever thought of the concept of angle as part of unlimited plane;
1 got the answers to questions 3b) and 3c) wrong, saying that Q did not belong to the angle and revealing in the interview that he didn't have a clear definition of angle.
He also got the answer to question 3e) wrong, saying that the two angles considered were not of the same size.
I.: Does point Q belong to angle 1?
1: No, the vertex and the sides belong to the angle, the point belongs to… that is, no, the size is equal both here and here, that is, I've never done the problem where a point belongs to an angle. We usually do the vertices and the sides of the angle. We don't do the points. Maybe now, we are discovering other things. The point belongs, huh!

3c)
I.: And in this case?
2: I think it should be the same thing, that is, not exactly the same thing, because the position is different.

3e)
I.: Are angles 1 and 2 the same size?
2: No.

1 did not understand the meaning of the term 'point belongs' with reference to an angle, being exclusively tied to the metric aspect of angle, that is, making the size and the angle coincide. However, this teacher proves consistent with himself; when he defines angle, he actually speaks about its size (even if in an incomplete way).

3b)
1: In what sense does it belong to angle 1?
I.: Is Q part of the angle that we have called 1?
1: (silence)
I.: Is Q a point of that angle, or not?
1: Uh, the angle you measure with the size, no? This will be… I don't understand the question. Roughly, you could say that the angle is always equal in size both here and here.
I.: But what is an angle?
1: The size of the angle is determined by the position of the two straight lines, from how they touch.
I.: So is Q part of this angle or not?
1: (silence)
I.: If I have a segment and I ask you if a point belongs to the segment… I mean a point is also a point of the segment besides being a point in itself (it represents the segment and indicates a point).
1: If it is on the segment yes, because it is part of that.
I.: Now, I ask you if Q is part of the angle which we have called 1, or not?
1: In what sense? I don't understand your question. In the sense if in this case there is an right angle [he points out a vertex of the table]…
I.: And if I ask you if this point is part of the right angle [he indicates a point on the table].
1: If to be part of means that it is on this surface, yes, but, however, the angle defined as

degree… I don't understand it as point.
I.: How will your pupils answer this question?
1: In my opinion, they will answer yes, because they see it inside. [Next question] And for P, it is clear that it is always inside of the angle, but they will be influenced by the sign of the angle which is up to here and P remains outside even if, in its own right, it is still inside the angle P.
I.: So you would have answered to question c) that P is still inside the angle; it belongs to the angle.
1: Yes, apart from belonging, in any case the angle is this, this is 90 degrees [he indicates the table again]. If I represent it on a sheet of paper and I do this or this [he indicates two different positions to measure] the angle is always the same.
I.: Therefore, this point here belongs as much to this as to this [he indicates points on the table]? Do they belong to the right angle?
1: Huh! Let's say yes [he wants to go on and continues with the next question]. They seem equal to me.

First situation – Didactic aspect

The didactic choices made by the teachers regarding the angle never involve showing possible misleading aspects of the concept of angle arising from graphic situations.
3: Not exactly graphics like this, but we have seen that the angle is not only the one marked, but the whole part between the two half lines.
I.: At the graphic level, do you introduce situations like the ones seen here?
3: No, to tell the truth, one of your students [Giada] did it. She had done a course on the introduction of angles and showed these things that, in any case, the points were inside.
I.: If you think that the representations are misleading in this way, why do you not use them didactically?
3: To tell the truth, I had never thought about it.

I.: Have you ever worked on the misleading aspects of the representations of angles?
5: No, to tell the truth, no, it should be done, I admit it.

I.: Do you show your pupils activities of this kind?
1: No, these I've never done, that is, it actually seems like the same angle, but the surface misleads… It's interesting! This indicates the size which here is smaller than the other; at first glance, for me, the children will be misled by this.

It is clearly noted that, in the classroom, the teachers speak mainly about angle as a measure, rather than from the conceptual and graphic point of view.
I.: How do you introduce the angle?
2: As a part of space, perhaps we can say that it belongs to the space. Honestly, we introduce the angle in daily life. They have to understand that it is definitely a part of space, then, that it has a vertex and sides which they must understand which can be lengthened, but we say that it belongs… I don't know. I don't remember.

I.: How do you introduce the angle?
2: I do it in real life, then we construct angles of different sizes or we give pre-made angles and have them find the size.
I.: So, it is a measurement activity.
2: Good for you, I have to admit that here we should broaden a bit, this yes; I admit that time constraints or deficiencies on our part mean that we should probably look at angles again. They know how to recognize them, but as representations sometimes, perhaps we could perhaps change a bit, sometimes we are very tied to measurements, also.

One teacher, however, stated preference, in class, for a representation to help the pupils remember that it isn't a limited figure (). He recognizes, in any case, that as a teacher, he often tends to always use the same representations which can create incorrect intuitive models.

The interview of one teacher showed how geometry tackled in class was reduced to minimal levels and that the concepts, such as the concept of angle, were not studied at all. In the fourth year, the image of angle that the pupils have is the one acquired in life, therefore very unrefined, empirical, and with all the potential misconceptions. In class, they had only classified the angles as acute, right, and obtuse. In year five they spoke of the angle of a polygon as part of a plane lying between the two sides. The teacher has never thought of the angle as an unlimited part of the plane and so has never mentioned it to his pupils. He recognizes that year 4 pupils in particular can have the idea of angle as a circular sector defined by the arc; part of a plane which the teacher usually has the pupils colour in. Moreover, he has never dealt with intersection of lines.

Comparison of expectations and class results

According to the teachers, several pupils will be misled by the questions proposed, as they were not part of their didactic practice. In this situation, there are fewer overestimated cases, while also some underestimated cases. In general, the forecasts are closer to the results.

I.: Will your pupils be misled or not?

3a)
3: In my opinion, yes, I already reason from the point of view that there is a reason for asking the question, but several make mistakes because they don't reason from the point of view that, if they ask you a question, there is… We have seen this argument, but for this question, I think many will make mistakes. Unless they use some means to deal with it, a set square.

3d)
3: It's the same as before, we have the half-lines that form two equal angles and two unequal angles, but the representation is very misleading. In my opinion, those using their acquired knowledge can answer correctly, but others will be misled by the graphic aspect.
I.: Will your pupils have the same problem?
2: I don't know. Maybe. Here, a well defined part is coloured, but it could be coloured here also. The angle measures here, measures here, the size is equal. It belongs, … huh! We have never placed a point that belongs.

1: We have never seen graphics quite like these, but we have seen that the angle is not only that which is marked, but all of the part included between the two half-lines so they should know it. Some, however, will say that Q yes [belongs] and P doesn't.
4: Yes, I admit it; my pupils will fall into this trap. I should have worked on it more.

3a)
1: In my opinion, they will say that angle 1 is larger.
I.: And is this true, in your opinion?
1: No, they will be misled by the dotted line part, but to me, they seem equal.

3d)
1: If they can understand and make the extension, it is like the earlier problem.
I.: So, in your opinion, your pupils will not be misled by the length of the sides of the angle, as a representation.
1: Some might, yes, but perhaps someone with a ruler could also extend the line, basically the half lines can also be extended.

In view of the small amount of geometry done in class (and the way in which it was presented), one teacher considered making any forecast excessively risky.

One teacher correctly predicts that his pupils would answer well to questions a) and b) (only in terms of Yes and No and not in terms of the justification), while for c) his evaluation is rather negative, "They will fall into the trap" which, however, is not confirmed by the result that wasn't so bad after all (50% correct answers). Many of the pupils' answers lack justifications, particularly for questions b) and c), or give justifications that are completely irrelevant (Does Q belong to angle 1? "Yes, because there would be degrees" or "because it is in the same square"), confirming the teacher's prediction.

Second situation – Conceptual aspect

Of the 6 teachers interviewed:
1 answered the questions correctly;
1 showed a situation of uncertainty about whether point C belonged to the highlighted angle;
When talking about the "internal" angles of a polygon, 1 teacher considered only the part belonging to the polygon, but, after having been alerted, returned to the part of the unlimited plane. It isn't, therefore, really a misconception, but a non-assimilated deeper understanding. So in didactic practice he had probably, perhaps unknowingly, induced this error in his pupils;
1 teacher made mistakes in the answers to questions 4b) and 4c), stating that the points do not belong to the angle, thinking of it as limited, even going against the chosen definition and, therefore, proving inconsistent.
I.: Does point C belong?

3: I say no, but what is the right answer?
I.: And P?
3: P no, P is outside.
I.: Why, in your opinion, is it called an internal angle?
3: Because it's inside the figure.
I.: What is an angle?
3: Part of a plane delimited by the two half-lines.
I.: Do you see half-lines in the polygon?
3: No, but in this case I see the sides.
I.: What do you see as unlimited half-lines? So, is the angle limited or unlimited?
3: No, but here it is limited because it is internal… Perhaps I should take a course, too.

1 teacher proved very uncertain in 4a) and 4c).

4a)
2: The point is in an internal angle this, no, meaning extending the sides of ...? If D belongs to the angle extending the sides, then, yes.
I.: So, for you, point D belongs to the angle.
2: No, no, but here they won't understand.
I.: You told me you're a pessimist.
2: But so much the better, if they surprise me, I've given my soul to these young people.

4b)
I.: And point C, on the other hand?
2: Point C is directly on the side, so it is part of the angle since it is on the side you consider it still inside, therefore it belongs to the side, you're teaching me.

4c)
I.: And point P, on the other hand?
2: If you do this [extends the sides], it is, yes, actually it is still inside, basically, even if here you have to truly see the figure on the plane with the extensions of the sides, otherwise you don't see it.
I.: Is it right to see the extensions of the sides?
2: Perhaps yes, here we should be even more... do activities on this and make the figures less static. I try to do it, but probably we should do it even more.

1 teacher didn't accept question 4). He only accepted that the points on the sides belonged to the angle and was undecided about 4c), still failing to understand the concept of belonging.

4a)
1: No.
I.: Why?
1: Because the angle is formed between BC and BA and this, only in this case [he indicates all the way up to the vertex of the side].

4b)
1: Returning to our earlier conversation, the answer is yes. It is its maximum and minimum part respectively.

4c)

1: Ah, good question. Yes, because if I extend here, it is inside. But then, also D… [he thinks]. I have a bit of a problem. Perhaps yes, I'm not sure, because I don't understand what you mean with this question.
I.: How do you deal with angle at school?
1: With them, I begin with practical things. We start them in the fourth year. The precise use of the goniometer is then consolidated in the fifth year. I begin by saying that a square has a right angle which measures 90 degrees, then they look around the room for right angles and they say a table, a corner of the blackboard, a book, a piece of furniture… I begin with the right angle which is easier and I let them draw it with a set-square on a sheet of paper and they revise and I explain that this is really only 90 degrees; not a degree more, not a degree less, while the difference with the other angles, I say that there are acute angles which I say measure from 0 degree up to 90, but 90 is already too much I say up to 89 degrees, in the fourth year I say this because they don't yet have the concept of decimal number at the beginning of the fourth year. The 90 degree right angle, then the straight angle, they draw a line and take the goniometer and then we look and there is also the round angle.
I.: So, you do a classification of the various kinds of angles.
1: Sometimes, I give them some exercises where they do it roughly without the goniometer and then we use the goniometer which isn't clear.
I.: Don't you show situations like the ones that I showed you?
1: No, I've never thought that I could tackle it like this. I wouldn't do the one with the point, but the rest is interesting, where you go a bit beyond, like with the fraction of the rectangle [he indicates the first question] where you go beyond the classic formula.

Only 1 teacher was uncertain about all the questions.

Second situation – Didactic aspect

No teacher shows his pupils the unlimitedness of an angle in a limited context such as polygons. Un limitedness is considered, not by all the teachers, only when presenting angle as a figure on its own. When polygons are addressed, they are usually presented as part of a plane defined by a line segment not as the intersection of angles. Therefore, for the pupil, the angle of a figure is limited by its sides. In any case, irrespective of the definition of polygon, there is no discussion of the unlimitedness of the angles in polygons. To confirm how, at the didactic level, discussion of angles is sometimes separated from that of plane figures, one teacher states, "Angle was done last year, this year they are working on polygons".

Generally, the teachers proved to be tied mainly to the measurement of space, and little to the conceptual and graphic aspect of angle.
I.: Do you ever show examples like these to your pupils?
5: No, I have to say no, also because we still haven't got to figures with them. We talk about angles. We measure the angles inside figures, but examples like this, no. I have never introduced them, not even in the fifth year.

One teacher stated that he never returns cyclically to topics in order to sort them out later on. For example, the concept of angle as part of an unlimited plane “is done at the beginning of the fourth year”, then never touched again.
The same teacher stated that intersecting lines had been dealt with only in the case of perpendicular half-lines. Particularly meaningful: the idea an unlimited part of the plane was not taken up again when using the goniometer. The pupils know that when they use this instrument, they can lengthen the sides, but this is treated as simply a “way of using” the goniometer.
4: This year, I did a bit of revision on the angle as part of a figure lying between two sides. I have never spoken about the unlimited angle. However, regarding the use of the goniometer, I said that sides that are too short can be extended.

Comparison of expectations and class results

In conclusion, according to the teachers interviewed, it is difficult to make forecasts about this situation, since the subject has not been addressed in class. The majority of teachers predicts negative results, which however, in several cases were overestimated.
I.: How will your pupils answer?
3: They will guess, since it’s a subject we haven’t dealt with.

In particular, one teacher correctly predicts that all the pupils will really struggle (at most 5-10% correct answers) because “for them, point D indicates another angle and point P is outside the figure”. In fact, only 15% answered that D belongs to the angle B, 25% for C, and 0% for P.

In three cases, at the end of the interview, the teachers had understood the kind of problem proposed, recognising the potential and stating that it would be a good idea to introduce them in the elementary school.
I.: Would you introduce an activity like this in class?
1: Yes, because it is interesting, because you get used to reasoning in a different way.

4.2.2 Middle school teachers

Here we report some considerations arising from the interviews with the 6 middle school teachers. It is interesting to note that the most unexpected answers were given by a teacher who trained and who has worked in the German part of Switzerland, and who is doing his first experiences at the Ticino school system.

With respect to the *conceptual aspect*, with reference to fractions, the middle school teachers were not perceived to encounter any particular problems in responding correctly to the questionnaire that was given to the pupils. The same cannot be said for angle. Uncertainty was noted in some teachers, and misconceptions in a few cases.
In terms of the *didactic aspect* with reference to the subject of fractions, however, all the MS teachers specified that they also introduce situations with non-congruent parts; stating their didactic value. The semiotic register chosen to represent fractions is usually the pictorial register, as observed even more clearly in the ES. There is not seen to be any particular didactic focus on preparing activities which aim to represent the same concept in different ways.
As regards the subject of *angles* and the didactic choices made by the MS teachers, they never consider the problem generated by possible misunderstandings of the concept of

angle deriving from the graphic situation. For angle, the teachers clearly usually present the aspect of its measure, rather than the conceptual aspect of figure. To the question, “How do you introduce angle to your pupils?”, most of the teachers answer, “I do it in the real world, then we construct angles of different sizes or we give them angles and have them find the size”.

4.2.2.1. Fractions

First situation – Conceptual aspect

The 6 teachers interviewed do not demonstrate any insecurities concerning the conceptual aspects presented in the first situation. They all show that the importance lies in the area, and not in the form. Their concept of fraction includes both the arithmetic and the geometric aspects.
We note that the MS Training Plan in Ticino (Middle School Training Plan, 2004) specifies that the concept of fraction as operator is dealt with in the second year. It is therefore not surprising that all the teachers answered the first three questions with certainty. Teacher 1 explicitly states that “[…] it is clear, you work with the fraction as operator on the area” thus confirming a clear conceptual vision of the subject.

First situation – Didactic aspect

Unlike the results reported for the ES teachers, the MS teachers maintain that pupils must be shown congruent parts, but they are aware of the importance of the non-congruent part approach. Despite this, only half of them have actually introduced didactic situations with non-congruent figures.
Three teachers (teachers 3, 5, and 6) have already dealt with the topic of fractions as operators in the first part of the school year, introducing examples with non-congruent subdivided figures. These teachers maintain that it is important and useful to show situations that are different from the classical one (with congruent parts). Since the pupils have already encountered similar situations, the teachers in question expect good results in this part of the questionnaire (for example, one teacher states “Most of the pupils should answer ‘yes’ to the three questions, justifying their answers with the fact that the figures have the same area, although the shapes are different. […] We have dealt with many similar examples with non-congruent figures. The rectangle is therefore the most commonly used figure […]”). One of these teachers has already addressed the addition of fractions and thinks that the strategy used by the pupils to answer the third question could be that of calculating ¼ + ¼, further complicating the situation. In fact, two students took this road: one concluded correctly that “1/4+1/4=2/4 reduced to the minimum terms is 1/2”. The other made a mistake stating that “1/4+1/4=2/8”.
One teacher (teacher 4) introduced the theme “fraction as operator” in the first year of middle school, but without presenting examples with figures subdivided into non-congruent parts. “I dealt with the subject last year, working with figures, segments or other sizes, however rarely with figures with different shapes”. He thinks that the pupils

could struggle with the first and third questions.
Teachers 1 and 2 had not revised the theme of fractions, but, in any case, they expect their pupils to produce positive results, because they maintain that these concepts are already acquired by the pupils in the elementary school.

Comparison of expectations and class results

In general, the teachers expected good results in all three items. Only two of them (4 and 5) predicted difficulties in the third item.
For the first two questions, this prediction proved to be justified for the answer by itself (without a stated reason). In fact, all the classes had a success rate of about 70%. The situation changes if you also take into account the explanations. In this case, the positive results go down, in one class even to 38%. This could be explained by the greater cognitive difficulty inherent in the request to give reasons for the answer.
The third question also produced a basically positive result of over 70%, except in the case of one class - for which the teacher did not predict any difficulty, despite not having worked with non-congruent figures - where we note a significant drop of about 90% in item 1.b to 57% in 1.c. On the other hand, the class results of the two teachers who had predicted difficulties were good (over 70%).

Second situation – Conceptual aspect

No particular problems related to teachers' beliefs emerged in this sphere. All the teachers interviewed stated with certainty that figures 1, 2, 3, 5, 6 represented the fraction ¾.
Figure 4 was seen as 3/7 by five teachers. Only one teacher (number 4) hesitated, but did not go as far as to state that figure 4 does not represent ¾.

The teachers preferred the first representation. Teacher 6 saw this partiality as a legacy from the elementary school. Teachers 1 and 5 stated that they were more used to representations of this kind. One teacher (4) even stated that, "It's clear that it is divided into 4 parts and 3 of them are taken". Teacher 3 also indicated representation 2 as his favourite. Teacher 2 was the only one to differ. In his opinion, the best representation was number 5, "since it is already written as ¾".

Second situation – Didactic aspect

The general tendency is to work with a reduced number of semiotic representations in order to avoid confusion.
The practice in Ticino leads one to prefer the first two representations, which seem very commonly used in the ES. This practice finds justification in the belief that it would be rather inappropriate to change the approach followed in the ES. Teacher 6 indirectly justified such a choice, referring to the Training Plan for mathematics in which representations different from 1 and 2 would never be explicitly indicated. He stated that "representations 5 and 6 are more for the third year of middle school", in the sphere of the subject of rational numbers, implicitly justifying the failure to address it in the second year. In fact, in the new Training Plan for the second year includes the specific objective "Recognising the fraction as a result of division […]".
It is interesting to note that teacher 2, who comes from outside Ticino and who indicated number 5 as the best representation, is the only one to nourish any perplexities about representation 1, even stating that such an image could evoke "[…] a 2x2 table".

The teachers thought that representation 3 was difficult to understand. Teacher 3 even asserts that, “It is the most difficult”. In fact, it was not always understood as a representation of the straight line of numbers. It was sometimes confused with the representation of the length of a segment. For example, in representation 3 teacher 5 did not see the straight line of numbers and therefore did not know how to associate it to ¾ as a number, but rather to ¾ as an operator applied to the length of the segment represented, stating that “[…] to be ¾ in the figure, the part equal to ¾ of the segment must be highlighted”.
Regarding representations 5 and 6, it is interesting to note how the practice described previously - which leads to separating the aspects of operator applied only to bi-dimensional geometric figures from the numeric ones - causes teachers 5 and 6 to predict that “[…] representations 5 and 6 will be placed in relationship to each other, but without connecting them to ¾”.

Comparison of expectations and class results

All of the teachers, except number 4, thought that the first (and possibly the second) depictions were the most chosen. This prediction was largely confirmed by the data.
Only teacher 6 explicitly expects positive results for representation 3, in the knowledge that he had worked on it. The data confirm the predictions of the teachers. We have a success rate of about 75% for the class of teacher 6 compared to an average of less than 25% for the others.
As expected, representation 4, recognised by all the teachers as unrelated to ¾, was not taken into consideration, not even by the pupils.
3 of them (1, 3, 4) hypothesised that representation 5 could also be one of those identified as ¾, due less to any true understanding of the concept, but rather because of a purely formal aspect. “[…] They could get there, if they think about the use of the “fraction” sign used in geometric formulae; for example, h=A/b=A:b”. The remaining 3 teachers predict that representation 5 will not be amongst those indicated, since the topic had not yet been addressed in class. Surprisingly, the data proved the predictions of all 6 teachers to be partially wrong. For those who hypothesised recognition, only slightly more than one quarter of the pupils actually recognised the representation. For those who hypothesised non-recognition, on the other hand, almost half of the pupils recognised it. The data also proved teachers 5 and 6 wrong; they predicted the association of representations 5 and 6, but separate from ¾.
Teacher 2 differs from the others. He thought that the pupils might see a table in the first illustration, and that representations 5 and 6 would clearly be chosen as the best representations of ¾. The data proved him wrong in that only a few pupils answered as he had predicted.

4.4.2.2 Angles

First situation – Conceptual aspect

The conceptual aspects of this situation are clear for 5 teachers. Teacher 2 admits to not knowing the definition of angle as part of a plane. For him, the angle "[…] is the measure of a rotation". However, all the teachers answered all the questions correctly, including teacher 2 after the definition of angle as a geometric figure was made clear to him.

First situation – Didactic aspect

In general, after being introduced in the first year of middle school, the concept of angle as a geometric figure is never used again. It is a sort of inert concept.
Their didactic practice favours operative aspects rather than conceptual aspects. None of the teachers interviewed has ever presented their students with problems concerning the angle as a set of points. All of them dedicate significant time to angle measurement, particularly to the use of the goniometer.
Opposite angles at the vertix, in particular the aspect relating to their congruence, are already addressed in the first year of middle school and the topic is applied to many problems.

Comparison of expectations and class results

All the teachers expected that the pupils "[…] will immediately ask to use the goniometer". They also thought that questions would generally be answered correctly. For them, the pupils "[…] except for a few" will answer correctly "[…] without being distracted by the arcs or by the extension of the coloured surfaces".
The analysis of the answers indicates:

for item a): two out of three pupils answered correctly, in most of the cases without justification; one of the most frequent justifications is the one related to measuring with a goniometer;

for items b) and c): the success rate of item b), more than 80%, coincided with the teachers' expectations; on the other hand, the success rate of item c) was half, dropping to a little more than 40%, clearly proving the teachers wrong; this highlights how the large number of correct answers in b) actually hides a misconception held by the pupils;

for items d) and e): in this case the teachers' predictions proved to be justified; approximately 3 out of 4 pupils answered both items correctly; the difference between the two situations did not seem to bother the pupils.

Second situation – Conceptual aspect

All the teachers answered correctly here as well (a couple with some hesitation), despite expressing amazement about questions of this type and stating more or less explicitly that they were seeing it for the first time; teacher 6 "[…] strange, I'd never thought about it".

Second situation – Didactic aspect

None of the teachers had ever introduced this kind of question; not so much because of an informed didactic choice, but rather because they had never considered the possibility of reflecting on similar aspects. Nevertheless, teacher 1 specified that he insists right from the start that the angle is unlimited.

Comparison of expectations and class results

Teachers 2, 3, and 4 did not manage to predict their pupils' answers. The other three made predictions, but only very prudent ones. Teacher 1 said that "[…] they should answer 'Yes', but there could be difficulties since D is the vertex of another angle" and he added that he often uses the expression "[…] angle in B, which could reinforce the idea that the angle is finished". Teacher 6 expected a majority of correct answers only for item a), while teacher 5 expected a majority of correct answers only for item b). Globally, the results for this second situation, were very low, only approaching 50% for item b). Only the class of teacher 1 obtained results bucking the trend, all between 50% and 60%, in line with his expectations.

5. Answers to the research questions

A1. Notwithstanding the studies conducted, the most diffuse beliefs amongst the elementary school teachers are similar to those held by their pupils; mainly with regard to the concept of angle, and to a lesser extent, fractions. Didactic habits, reference to one's own experience and that of colleagues, reference to texts or to specific materials used in Ticino, very often reaffirm that the knowledge possessed by teachers greatly influences what they demand of their pupils. One does not see a real didactic transposition understood as a re-elaboration of Knowledge (D'Amore, 1999b). It seems that the teacher limits himself to providing his own knowledge to his pupils. Therefore, the cognitive and didactic points of view are not far apart; there is a tendency to identify the belief about a concept with the way in which it is introduced, with the way with which it is cognitively understood. The situation is slightly different for the middle school teachers, some of whom have a deeper understanding of the concepts in question. However, this culture does not modify the didactic methods, so even in this case, preference is given to executing a minimum didactic transposition, almost identifying the cognitive and didactic aspects.

A2. There is no difference between the two scholastic levels. The rules implicit in the didactic behaviour associated with the teacher's epistemology (Brousseau, 2008) come into play and the teacher maintains that the performance of the pupil should coincide with the repetition of the activities (oral and written) which have marked his teaching action. The expected performances are rather superficial and repetitive. In fact, faced with situations defined as 'new' within our proposal on this theme, there are opposite, but cognitively similar, reactions: a) these proposals cannot enter the didactics because they would change the scholastic programme; b) I had never though about these proposals and I will use them in the future. Both reactions demonstrate that teachers' expectations tend not to enter into Vygotsky's 'zone of proximal development', but to limit themselves to the effective zone.

A3. A double answer is given to the first question of Q3; one general and one specific. The general answer: as with all other mathematics topics, fractions and angles are also intrinsic concepts in the scholastic programme, for which the teacher gives a definition which must be accepted (on the whole identical for both the elementary and the middle schools), the learning of which is not important. The important thing is to tell the teacher what he expects to hear. The specific answer: angle and fraction are subjects which are not encountered in the extra-scholastic reality and which are supplied only with information, explicit or implicit, given in the classroom. The answer to the second question of Q3 is formulated on the basis of the observation-comparison of the responses given in the elementary school and in the middle school. If we compare the two school levels, we do not see any significant differences between students' beliefs regarding angles and fractions. Notwithstanding the age and maturity differences, the mental model formed in the elementary school remains in the middle school and the apparently paradoxical situations proposed in order to initiate discussion are perceived as such at both scholastic levels.

A4. The answer to Q4 is positive; in the sense that the teachers do not seem surprised if a student gives a wrong answer, or in any case an answer not coherent with the definitions. The teachers, adult and educated, seldom fail to see the "deception" in some of our questions. However, when analyzing the test, they tend to recognise the answers of their own pupils.

A5. To answer Q5, it is necessary to make a significant distinction between the two scholastic levels. In the elementary school, as said previously, the knowledge of teachers and the expected knowledge of students are virtually identical. In the middle school, as also said previously, the teachers possess a greater mathematical conceptual education, but they do not submit their own knowledge to a true didactic transposition. Nevertheless, although this knowledge nourishes different conceptual beliefs, it does not greatly alter the expectations of the teachers at the two scholastic levels. Their expectations of these mathematical subjects are rather similar. So for different reasons, the beliefs of the teachers influence the beliefs matured by the students; misconceptions included.

A6. The history of mathematical thought demonstrates, without a shadow of a doubt, that angle and fraction are concepts for which one can speak about epistemological obstacle. (D'Amore, 1985; D'Amore, Marazzani, 2008; Fandiño Pinilla, 2005, chap. 2). However, the analysis conducted shows that, precisely because of the choices (which for the elementary school teachers are obligatory, while for the middle school teachers they are due to their beliefs about the abilities and possibilities of their pupils), the epistemological obstacles are superimposed by didactic obstacles. As we have demonstrated, these didactic obstacles are tied to the beliefs of teachers regarding the two mathematical concepts and the learning of their pupils.

6. Conclusions

The results demonstrate that, in some cases, the cognitive and didactic points of view of the teachers are not in any way distinct from each other. There is a tendency to identify

the belief about a concept with the way in which it is proposed and with the way in which it is cognitively expected. The situation is slightly different for the middle school teachers, some of whom have a deeper understanding of the concepts in question. However, this culture does not always modify the didactic methods for angles, so even in this case one ends up by executing an ineffective didactic transposition.
The beliefs of the teachers, both cognitive and didactic, precisely define the classroom activities and also influence their interpretation of their role; what to teach, how and why.

The preliminary beliefs of the teacher also run the relationship in the classroom. The teaching of correct mathematics is not important, and therefore neither is hypothesising correct learning process. It is more important to obtain the answers that were expected right from the beginning of the activity.
The questionnaires proposed by us include questions chosen purposely in order to highlight the fact that even students who seem to have conceptualised a concept well, in fact limit themselves to good contractual performance: they answer the expected questions, probably because of the "Topaz" effect (D'Amore, Fandiño Pinilla, 2008) or similar.
The most striking example involves the angle; even having admitted and repeated in response to an explicit question that the angle is a certain unlimited part of a plane, in fact, the addition of the adjective "internal" referring to polygons profoundly modifies conceptualisation, leading to situations expressing explicit contradictions.
The results therefore demonstrated that the conceptualisation of these two concepts, fractions and angles, *did not always happen*, particularly as regards the angle. There was no real construction of knowledge; not only from a conceptual aspect, but not even from a semiotic aspect, because the different semiotic representations of the angle modify their conceptualization.
Amongst the many possible causes, our research clearly shows the influence exerted by teachers beliefs, which, as has been seen, determine both the beliefs of the students, and in some cases their failure to develop significantly over time, and also the teachers' fear of proposing sufficiently varied and rich situations, adding didactic obstacles (avoidable to (objective) epistemological obstacles.

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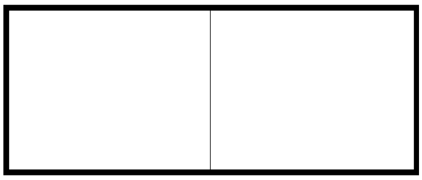
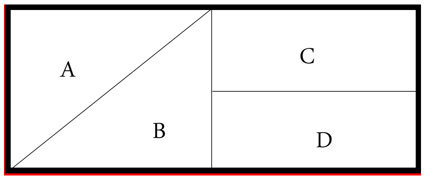
Attachment. Pupils' questionnaire

FRACTIONS of a continuous unit

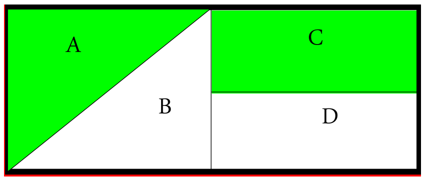
Here is a rectangle divided into two parts.



Now, each of the two parts is divided into two equal parts.
So, there are four parts: A, B, C, D.



Two of these parts have been coloured in.



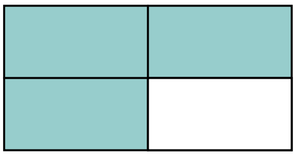
Does part A represent $\frac{1}{4}$ of the first rectangle? Yes ☐ No ☐ Why?

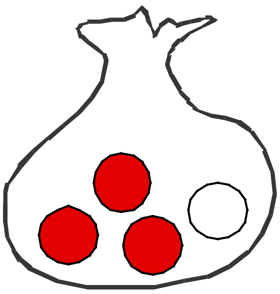
Does part C represent $\frac{1}{4}$ of the first rectangle? Yes ☐ No ☐ Why?

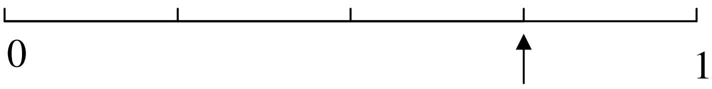
Does all of the coloured part represent $\frac{1}{2}$ of the first rectangle? Yes ☐ No ☐ Why?

Representation of FRACTIONS

Look at the following figures, diagrams, and writing.
a) Beside each representation, write what it is trying to communicate.

1. 

2. 

3. 

4. 3 : 4

5. 0,75

b) Of these preceding five representations, which of them say the same thing? Why?

c) In your opinion, does one of them 'better' represent the fraction $\frac{3}{4}$? Yes ☐ No ☐

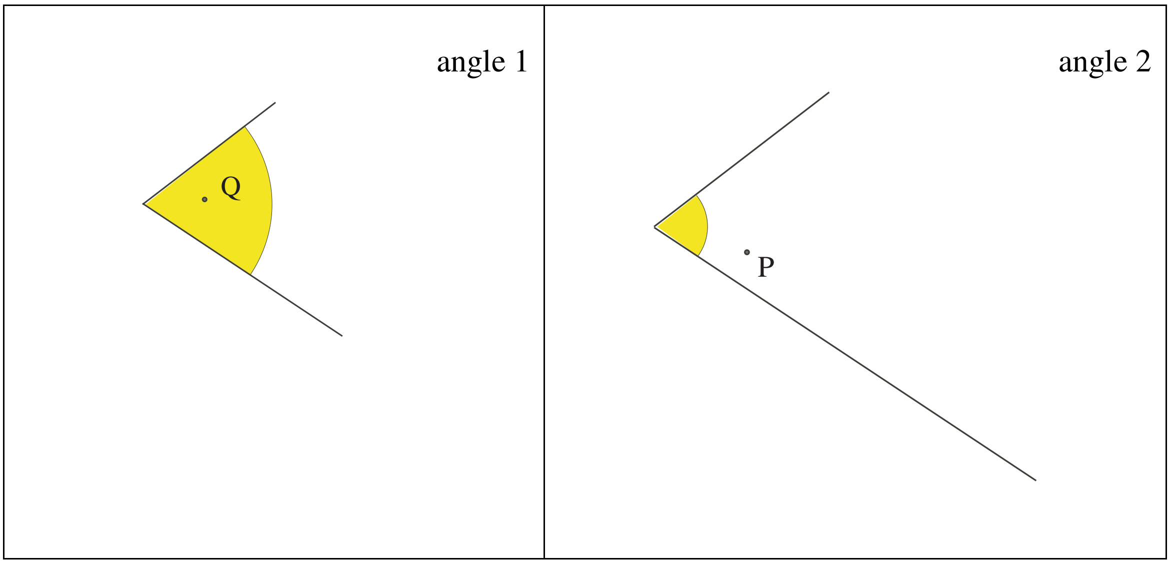
If yes, which and why?

d) Are there any that do not represent the fraction $\frac{3}{4}$? Yes ☐ No ☐

If yes, which and why?

ANGLES

Look at the two angles represented below, which we have called angle 1 and angle 2:



Answer the following questions:

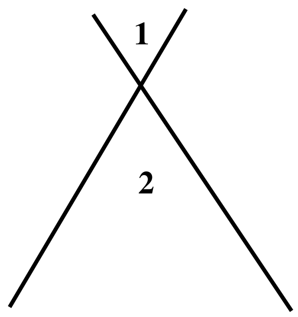
a) Between the two angles 1 and 2, is there one that is a larger size than the other?

Yes ☐ No ☐ Why?

b) Does point Q belong to angle 1? Yes ☐ No ☐ Why?

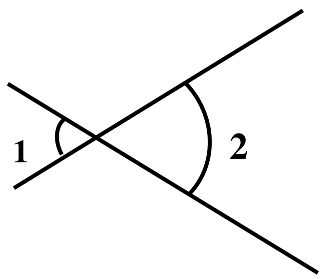
c) Does point P belong or not to angle 2? Yes ☐ No ☐ Why?

d) Two straight lines intersect forming the acute angles 1 and 2.



Is the size of one of these larger? Yes ☐ No ☐ Why?

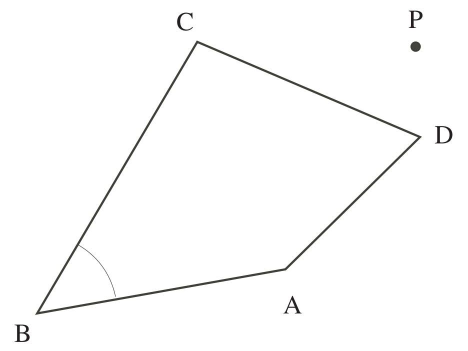
e) Two straight lines intersect forming the acute angles 1 and 2.



Is the size of one of these larger? Yes ☐ No ☐ Why?

INTERNAL ANGLES

In the quadrilateral ABCD, an internal angle has been highlighted.



Answer the following questions:

a) Does point D belong to the highlighted angle? Yes ☐ No ☐ Why?

b) Does point C belong to the highlighted angle? Yes ☐ No ☐ Why?

c) Does point P belong to the highlighted angle? Yes ☐ No ☐ Why?

Questions of the type posed during the teacher interview

Fractions

Have you seen the test that we have proposed? In your opinion, what will your more able pupils answer? Why?

And the ones that are less able? Why?

How do you usually introduce the concept of fractions?

Have you ever proposed examples of this kind? Why? Which ones do you usually show?

Does the shape of the highlighted parts affect the answer? From what point of view?

With reference to the representation of fractions, since it deals with different semiotic representations of the same concept, they should be equi-meaningful with each other. However, certainly for the children it will not be like that. Do you use all of them?

Which do you maintain are the most advantageous?

In your opinion, which one or ones will the children choose?

Do you have in mind others that you would propose to your pupils?

Angles

Have you seen the test that we have proposed? In your opinion, how will your more able pupils answer? Why?

And the less able? Why?

How do you usually introduce the concept of angle?

And how do you handle it after that?

Do you propose situations of this kind?

Yes / No, why?